CONSTRUCTION AND ANALYSIS OF $2q \times 2^2$ ASYMMETRICAL FACTORIAL DESIGNS IN TWO REPLICATIONS

By M. G. Sardana*

Institute of Agricultural Research Statistics, New Delhi

1. Introduction

The construction and analysis of confounded designs for asymmetrical factorial experiments have become fairly simple on account of the recent work done by Kishen and Srivastava (1959) and Das (1960). One of the methods given by Kishen and Srivastava (1959) is for the construction of the designs for asymmetrical factorial experiments of the type $q \times 2^2$ in $2q$ plot blocks by the use of associate B.I.B. designs. Kishen (1960), and Kishen and Tyagi (1961), following Shah's (1960) technique, gave further methods of construction of these designs by splitting the associate B.I.B. design into two P.B.I.B. designs and making use of one of them. The blocks of the design, constructed in this manner, cannot be divided into groups such that each group forms a complete replication. Also, the design constructed by the methods described by Kishen and Srivastava (1959), Kishen (1960), and Kishen and Tyagi (1961) require large number of replications. Another important presumption in the construction of the designs by either of these methods is the existence of a B.I.B. design with parameters $V = q, b, k, r, \lambda$. This may not be always true.

Since economy in the use of resources in experimentation is very essential, it is desirable to construct designs which need relatively small amount of resources and at the same time provide mutually independent estimates of all the effects including those affected by block differences. The object of the present paper is to give the methods of construction and analysis of designs for asymmetrical factorial experiments of the type $2q \times 2^2$ in $4q$ plot blocks involving only two replications, which provide mutually independent estimates of all the effects.

The construction and analysis of the designs will be discussed separately for the two cases—(i) q is even and (ii) q is odd.

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* Present address: Central Potato Research Institute, Simla.

2. Design and Analysis for q Even

If q is even, then design $2q \times 2^2$ in $4q$ plot blocks will be of the type $4q' \times 2^2$ in $8q'$ blocks, where $q = 2q'$. Let the factors and their levels be denoted as under:

Factor	*Levels*
A	$a_0, a_1, a_2, \ldots, a_{4q'-1}$
B	b_0, b_1
C	c_0, c_1.

To construct this design, we introduce a pseudo-factor α at two levels α_0 and α_1, where α_0 denotes the combinations b_0c_0, b_1c_1 and α_1 the combinations b_0c_1, b_1c_0 of the factors B and C. In the first place we construct a design for $4q' \times 2$ in two replications and $4q'$ plot blocks involving the factors A and α. Two replications of this design can be obtained by confounding any one d.f. out of the $(4q' - 1)$ d.f. of the interaction $A\alpha$ in one replication and another d.f. of the interaction $A\alpha$, orthogonal to the first one, in the second replication. For instance, the two orthogonal confounded components of the interaction $A\alpha$ in the two replications may be taken as:

$$(a_0+a_1+a_2+\ldots+a_{2q'-1} - a_{2q'} - a_{2q'+1} \ldots - a_{4q'-1})(\alpha_0 - \alpha_1)$$

and

$$(a_0 - a_1+a_2 -+\ldots - a_{2q'-1}+a_{2q'} -+\ldots - a_{4q'-1})(\alpha_0 - \alpha_1)$$

respectively. This would result in a design in which the first block of the first replication contains combinations of α_0 with levels a_0, a_1, $a_2, \ldots, a_{2q'-1}$ of the factor A and α_1 with the other $2q'$ levels of the factor A, while the second block of the first replication can be obtained from the first block by simply interchanging α_0 and α_1. Similarly the contents of the blocks of the second replication can be written out. Replacing α_0 and α_1 by b_0c_0, b_1c_1; and b_0c_1, b_1c_0 respectively, the required design for $4q' \times 2^2$ in $8q'$ plot blocks can be obtained.

It is evident from the method of construction that the only interaction affected by block differences in the designs constructed by the above method is the three-factor interaction ABC, for which the loss of information on each of the two confounded d.f. is $\frac{1}{2}$, while for the remaining $(4q' - 3)$ d.f. the loss of information is zero. The analysis of such designs is straightforward—the unconfounded effects being computed in the usual way, while the confounded effects are calculated only from the replication in which they are free from block effects.

3. Design and Analysis for q Odd

3.1. *Design.*—When q is odd, the method indicated in Section 2 cannot be followed because it is not possible to put down two orthogonal contrasts of the interaction $A\alpha$, which involve all the $2q$ levels of the factor A with coefficients either $+1$ or -1. Thus, when q is odd, some other technique is to be developed to construct the design. In the first place we construct the design $2q \times 2$ in $2q$ plot blocks and two replication involving the factors A and α. The first block of the first replication is obtained by combining α_0 level of the pseudo-factor α with any q levels of the factor A and α_1 level of the pseudo-factor α with the remaining q levels of the factor A. The contents of the second block can be obtained from the first block by interchanging α_0 and a_1. The q levels of the factor A which occur with a_0 level of the factor α in the first block may be called the generator of the first replication. The second replication of the design can similarly be obtained by starting with a generator different from the one used in the first replication. These two replications provide the design $2q \times 2$ in $2q$ plot blocks. Replacing α_0 and α_1 by b_0c_0, b_1c_1; and b_0c_1, b_1c_0 respectively, the required design for $2q \times 2^2$ in $4q$ plot blocks can be written out.

3.2. *Analysis.*—It is evident from the method of construction that the only interaction affected by block differences in these designs is the interaction ABC. Using the factorial model, the confounded interaction effect ABC ($2q - 1$ d.f.) can be estimated by solving a set of normal equations obtained through the method of least squares. ABC ($2q - 1$ d.f.) can be defined as $(2q - 1)$ independent comparisons between Z_i's ($i = 0, 1, 2, \ldots, \overline{2q-1}$), where $Z_i = a_i(\alpha_0' - \alpha_1)$. If the generators for the first and second replications are chosen as a_0, a_1, a_2, a_3, $\ldots$, a_{q-1}; and a_1, a_2, a_3, $\ldots$, a_q respectively, then it can be shown that out of the $(2q - 1)$ d.f. for the interaction ABC, only two d.f. represented by the following contrasts are affected by block differences:

(i) Contrast I $\rightarrow (Z_0 - Z_q)$

and

(ii) Contrast II $\rightarrow \left(\sum_{i=1}^{q-1} Z_i - \sum_{i=q+1}^{2q-1} Z_i\right)$.

The estimates of the confounded contrasts, adjusted for blocks, are given by:

$$\text{Estimate for contrast I} = \frac{2q}{16(q-1)}[Q_0 - Q_q]$$

$$\text{Estimate for contrast II} = \frac{2q}{16}\left[\sum_{i=1}^{q-1} Q_i - \sum_{i=q+1}^{2q-1} Q_i\right].$$

where

$$Q_i = T_i - 2(B)_i/4q,$$

T_i = Sum of the observations from the treatment combinations $a_ib_1c_1$ and $a_ib_0c_0$ *minus* the sum of the observations from the treatment combinations $a_ib_0c_1$ and $a_ib_1c_0$.

$(B)_i$ = Sum of those block totals which contain the i-th level of the factor A with α_0 *minus* the sum of those block totals which contain the i-th level of the factor A with α_1.

The confounded contrasts can be seen to have a direct correspondence with the elements of the generators. Contrast I is seen to correspond to the elements which are different between the two generators, *viz.*, a_0 and a_q, while contrast II corresponds to the comparison of the $(q-1)$ elements common between the generators with the remaining $(q-1)$ levels of the factor A, *viz.*, $(a_1, a_2, a_3, \ldots, a_{q-1})$ *versus* $(a_{q+1}, a_{q+2}, \ldots, a_{2q-1})$.

The sum of squares due to the $(2q-3)$ unconfounded d.f. of ABC is given by

$$\left\{\frac{1}{8}\sum_{i=0}^{2q-1} T_i^2 - \frac{1}{16q}\left(\sum_{i=0}^{2q-1} T_i\right)^2\right\}$$

$$-\left\{\frac{1}{16}(T_0 - T_q)^2 + \frac{1}{16(q-1)}\left[\sum_{i=1}^{q-1} T_i - \sum_{i=q+1}^{2q-1} T_i\right]^2\right\}.$$

The sum of squares due to contrasts I and II are given by

$$\frac{2q}{32(q-1)}[Q_0 - Q_q]^2 \quad \text{and} \quad \frac{2q}{32(q-1)}\left[\sum_{i=1}^{q-1} Q_i - \sum_{i=q+1}^{2q-1} Q_i\right]^2$$

respectively.

The relative loss of information on contrasts I and II is $1/q$ and $(q-1)/q$ respectively. Thus the total relative loss of information is 1; the number of d.f. confounded per replication.

4. Summary

Designs providing mutually independent estimates of all the effects in experiments of the type $2q \times 2^2$ in $4q$ plot blocks have been obtained in only two replications. The use of these designs will demand sufficiently lesser amount of resources for experimentation as compared to designs already available in the literature.

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6. References

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